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Subdivision of the Spectra for Difference Operator over Certain Sequence Space

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ABSTRACT

In a series of papers, B. Altay, F. Başar and A. M. Akhmedov recently investigated the spectra and fine spectra for difference operator, considered as bounded operator over various sequence spaces. In the present paper approximation point spectrum, defect spectrum and compression spectrum of difference operator Δ over the sequence spaces c_0, c, ℓ_p and bv_p are determined, where bv_p denotes the space of all sequences (x_k) such that $(x_k - x_{k-1})$ belongs to the sequence space ℓ_p and $1 < p < \infty$.

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1. PRELIMINARIES, BACKGROUND AND NOTATION

Let X and Y be the Banach spaces and $T : X \to Y$ also be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.,

$$R(T) = \{y \in Y : y = Tx, x \in X\}.$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$ then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^* f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

Let $X \neq \{\theta\}$ be a non trivial complex normed space and $T : D(T) \to X$ a linear operator defined on subspace $D(T) \subseteq X$. We do not assume that $D(T)$ is dense in X, or that T has closed graph $\{(x,Tx) : x \in D(T)\} \subseteq X \times X$. We mean by the expression "T is *invertible*" that there exists a bounded linear operator $S : R(T) \to X$ for which $ST = I$ on $D(T)$ and $R(T) = X$; such that $S = T^{-1}$ is necessarily uniquely determined, and linear; the boundedness of S means that T must be *bounded below,* in the sense that there is $k > 0$ for which $\|Tx\| \geq k\|x\|$ for all $x \in D(T)$. Associated with each complex number λ is perturbed operator

$$T_\lambda = \lambda I - T,$$

defined on the same domain $D(T)$ as T. The *spectrum* $\sigma(T,X)$ consist of those $\lambda \in \mathbb{C}$ for which T_λ is not invertible, and the *resolvent* is the mapping from the complement $\sigma(T,X)$ of the spectrum into the algebra of bounded linear operators on X defined by $\lambda \mapsto T_\lambda^{-1}$.

2. SUBDIVISION OF THE SPECTRUM

In this section, we mention from the parts point spectrum, continuous spectrum, residual spectrum, approximate point spectrum, defect spectrum and compression spectrum of the spectrum. There are many different ways to subdivide the spectrum of a bounded linear operator; some of them are motivated by applications to physics, in particular, quantum mechanics.

2.1. The point spectrum, continuous spectrum and residual spectrum.

The name *resolvent* is appropriate, since T_λ^{-1} helps to solve the equation $T_\lambda x = y$. Thus, $x = T_\lambda^{-1} y$ provided T_λ^{-1} exists. More important, the investigation of properties of T_λ^{-1} will be basic for an understanding of the operator T itself. Naturally, many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all λ's in the complex plane such that T_λ^{-1} exists.

Boundedness of T_λ^{-1} is another property that will be essential. We shall also ask for what $\lambda's$ the domain of T_λ^{-1} is dense in X, to name just a few aspects. A *regular value* λ of T is a complex number such that T_λ^{-1} exists and bounded, and whose domain is dense in X. For our investigation of T, T_λ and T_λ^{-1}, we need some basic concepts in spectral theory which are given as follows (see Kreyszig (1978)).

The *resolvent set* $\rho(T,X)$ of T is the set of all regular values λ of T. Furthermore, the spectrum $\sigma(T,X)$ is partitioned into three disjoint sets as follows:

(i) The *point (discrete) spectrum* $\sigma_p(T,X)$ is the set such that T_λ^{-1} does not exist. An $\lambda \in \sigma_p(T,X)$ is called an *eigenvalue* of T.

(ii) The *continuous spectrum* $\sigma_c(T,X)$ is the set such that T_λ^{-1} exists and is bounded and the domain of T_λ^{-1} is dense in X.

(iii) The *residual spectrum* $\sigma_r(T,X)$ is the set such that T_λ^{-1} exists (and may be bounded or not) but the domain of T_λ^{-1} is not dense in X.

Therefore, these three subspectras form a disjoint subdivision

$$\sigma(T,X) = \sigma_p(T,X) \cup \sigma_c(T,X) \cup \sigma_r(T,X). \tag{1}$$

To avoid trivial misunderstandings, let us say that some of the sets defined above, may be empty. This is an existence problem which we shall have to discuss. Indeed, it is well-known that $\sigma_c(T,X) = \sigma_r(T,X) = \emptyset$ and the spectrum $\sigma(T,X)$ consists of only the set $\sigma_p(T,X)$ in the finite dimensional case.

2.2. The approximate point spectrum, defect spectrum and compression spectrum.

In this subsection, following Appell *et al.* (2004), we give the definitions of the three more subdivisions of the spectrum called as the *approximate point spectrum, defect spectrum* and *compression spectrum.*

Given a bounded linear operator T in a Banach space X, we call a sequence (x_k) in X as a *Weyl sequence* for T if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$, as $k \to \infty$.

In what follows, we call the set

$$\sigma_{ap}(T, X) := \{\lambda \in \mathbb{C} : \text{ there exists a Weyl sequence for } \lambda I - T\} \tag{2}$$

the *approximate point spectrum* of T. Moreover, the subspectrum

$$\sigma_{\delta}(T, X) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\} \tag{3}$$

is called *defect spectrum* of T.

The two subspectra given by (2) and (3) form a (not necessarily disjoint) subdivision

$$\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_{\delta}(T, X) \tag{4}$$

of the spectrum. There is another subspectrum,

$$\sigma_{co}(T, X) = \left\{\lambda \in \mathbb{C} : \overline{\mathbb{R}(\lambda I - T) \neq X}\right\},$$

which is often called *compression spectrum* in the literature and which gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_{co}(T, X)$$

of the spectrum.

Clearly, $\sigma_p(T, X) \subseteq \sigma_{ap}(T, X)$ and $\sigma_{co}(T, X) \subseteq \sigma_{\delta}(T, X)$. Moreover, comparing these subspectra with those in (1) we note that

$$\sigma_r(T, X) = \sigma_{co}(T, X) \setminus \sigma_p(T, X)$$

and

$$\sigma_c(T,X)=\sigma(T,X)\setminus\left[\sigma_p(T,X)\cup\sigma_{co}(T,X)\right].$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint. Building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints.

Proposition 2.1. (Appell (2004)). *Spectra and subspectra of an operator* $T\in B(X)$ *and its adjoint* $T^*\in B(X^*)$ *are related by the following relations:*

(a) $\sigma(T^*,X^*)=\sigma(T,X)$.

(b) $\sigma_c(T^*,X^*)\subseteq\sigma_{ap}(T,X)$.

(c) $\sigma_{ap}(T^*,X^*)=\sigma_\delta(T,X)$.

(d) $\sigma_\delta(T^*,X^*)=\sigma_{ap}(T,X)$.

(e) $\sigma_p(T^*,X^*)=\sigma_{co}(T,X)$.

(f) $\sigma_{co}(T^*,X^*)\supseteq\sigma_p(T,X)$.

(g) $\sigma(T,X)=\sigma_{ap}(T,X)\cup\sigma_{ap}(T^*,X^*)=\sigma_p(T,X)\cup\sigma_{ap}(T^*,X^*)$.

The relations (c)-(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum dual to the compression spectrum.

The case (g) implies, in particular, that $\sigma(T,X)=\sigma_{ap}(T,X)$ if X is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces (see (Appell (2004)).

2.3. Goldberg's classification of spectrum.

If X is a Banach space and $T\in B(X)$, then there are three possibilities for $R(T)$:

(I) $R(T) = X$.

(II) $R(T) \neq \overline{R(T)} = X$

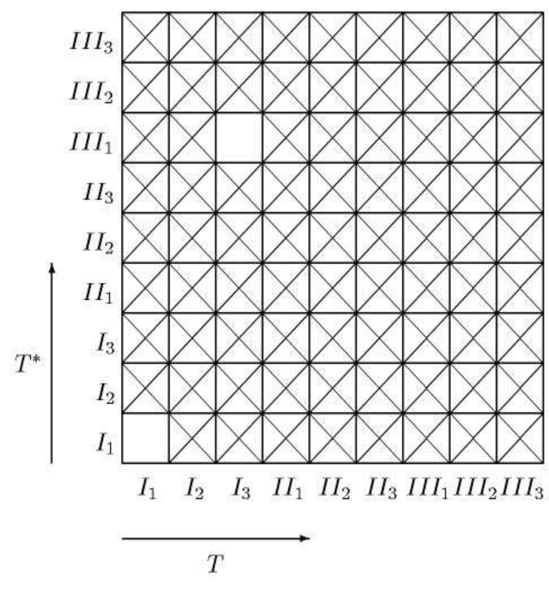
(III) $\overline{R(T)} \neq X$.

and

(1) T^{-1} exists and is continuous.
(2) T^{-1} exists but is discontinuous.
(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labeled by $I_1, I_2, I_3, II_1, II_3, III_1, III_2, III_3$. If an operator is in state III_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exist but is discontinuous (see Goldberg (1966)).

TABLE 1: State diagram for $B(X)$ and $B(X^*)$ for a non-reflective Banach space X



If λ is a complex number such that $T_\lambda = \lambda I - T \in I_1$ or $T_\lambda = \lambda I - T \in II_1$, then $\lambda \in \rho(T, X)$. All scalar values of λ not in $\rho(T, X)$. comprise the spectrum of T. The futher classification of $\sigma(T, X)$ gives rise to the fine spectrum of T.

That is, $\sigma(T,X)$ can be divided into the subsets $I_2\sigma(T,X)=\emptyset, I_3\sigma(T,X), II_2\sigma(T,X), II_3\sigma(T,X), III_1\sigma(T,X), III_2\sigma(T,X)$, $III_3\sigma(T,X)$. For example, if $T_\lambda^{-1}=\lambda I-T$ is in a given state, III_2 (say), then we write $\lambda\in III_2\sigma(T,X)$.

By the definitions given above, we can illustrate the subdivision (2.1) in the following table.

TABLE 2: Subdivisions of spectrum of a linear operator

		1	2	3
		T_λ^{-1} exists and is bounded	T_λ^{-1} exists and is unbounded	T_λ^{-1} does not exists
I	$R(\lambda I-T)=X$	$\lambda\in\rho(T,X)$		$\lambda\in\sigma_p(T,X)$ $\lambda\in\sigma_{ap}(T,X)$
II	$\overline{R(\lambda I-T)}=X$	$\lambda\in\rho(T,X)$	$\lambda\in\sigma_c(T,X)$ $\lambda\in\sigma_{ap}(T,X)$ $\lambda\in\sigma_\delta(T,X)$	$\lambda\in\sigma_p(T,X)$ $\lambda\in\sigma_{ap}(T,X)$ $\lambda\in\sigma_\delta(T,X)$
III	$\overline{R(\lambda I-T)}\neq X$	$\lambda\in\sigma_r(T,X)$ $\lambda\in\sigma_\delta(T,X)$ $\lambda\in\sigma_{co}(T,X)$	$\lambda\in\sigma_r(T,X)$ $\lambda\in\sigma_{ap}(T,X)$ $\lambda\in\sigma_\delta(T,X)$ $\lambda\in\sigma_{co}(T,X)$	$\lambda\in\sigma_p(T,X)$ $\lambda\in\sigma_{ap}(T,X)$ $\lambda\in\sigma_\delta(T,X)$ $\lambda\in\sigma_{co}(T,X)$

Observe that the case in the first row and second column cannot occur in a Banach space X, by the closed graph theorem. If we are not in the third column, i.e., if λ is not an eigenvalue of T, we may always consider the resolvent operator T_λ^{-1} (on a possibly "thin" domain of definition) as "algebraic" inverse of $\lambda I-T$.

By a *sequence space,* we understand a linear subspace of the space $\omega=\mathbb{C}^{\mathbb{N}_1}$ of all complex sequences which contains ϕ, the set of all finitely non-zero sequences, where $\mathbb{N}_1$ denotes the set of positive integers. We write ℓ_∞, c, c_0 and bv for the spaces of all bounded, convergent, null and bounded variation sequences, respectively. Also by ℓ_p, we denote the space of all p-absolutely summable sequences, where $1\leq p<\infty$.

In this paper, our main focus is the difference operator Δ represented by the matrix

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We give the subdivisions of the spectrum of the matrix Δ on the spaces c_0, c, ℓ_p and bv_p, where $1 < p < \infty$.

3. THE SUBDIVISONS OF THE SPECTRUM OF THE MATRIX Δ ON THE SPACES c_0, c, ℓ_p AND bv_p

In 2004, Altay and Başar (2004) determined the spectra and the fine spectra of difference operator Δ on the sequence spaces c_0 and c. In 2006, Akhmedov and Başar (2006) determined the spectra and the fine spectra of difference operator Δ on the space ℓ_p, where $1 \leq p \leq \infty$. In 2007, Altay and Başar (2007) determined the spectra and the fine spectra of difference operator Δ on the space ℓ_p, where $0 < p < 1$. In 2007, Akhmedov and Başar (2006) determined the spectra and the fine spectra of difference operator Δ on the space bv_p, where $1 \leq p < \infty$ and bv_p denotes the space of sequences of p-bounded variation introduced by Başar and Altay (2003) consisting of all sequences $x = (x_k)$ such that $(x_k - x_{k-1}) \in \ell_p$. In 2006, Kayaduman and Furkan (2006) determined the spectra and the fine spectra of difference operator Δ on ℓ_1 and bv. In this paper we developed the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator Δ over the sequence spaces c_0, c, ℓ_p and bv_p, where $1 < p < \infty$.

3.1. Subdivision of the spectrum of Δ on c_0.

We give the subdivisions of the spectrum of the difference operator Δ over the sequence space c_0.

Lemma 3.1. $III_2\sigma(\Delta,c_0)=\{\lambda:|\lambda-1|<1\}/\{1\}$.

Proof. By Theorem 2.5 of Altay and Başar (2004), $\Delta-\lambda I\in III_1\cup III_2$. Since $\Delta-\lambda I$ is triangle, $(\Delta-\lambda I)^{-1}$ exists. Then, by solving $(\Delta-\lambda I)x=y$ for x in terms of $\boldsymbol{y}$ gives the matrix $(\Delta-\lambda I)^{-1}$. The nth row turns out to be

$$\begin{cases} 0 & ,k>n \\ \dfrac{(1-\lambda)^k}{(1-\lambda)^{n+1}} & ,k\le n \end{cases}$$

Thus, we observe that

$$\left\|(\Delta-\lambda I)^{-1}\right\|=\sup_{n\in\mathbb{N}_1}\sum_k\frac{|1-\lambda|^k}{|1-\lambda|^{n+1}}. \tag{5}$$

Hence, by (5), the inverse of the operator $\Delta-\lambda I$ is discontinuous. Therefore, $\Delta-\lambda I$ has an unbounded inverse. □

Corollary 3.2. $III_1\sigma(\Delta,c_0)=\{1\}$.

Proof. By Theorem 2.5 of Altay and Başar (2004),

$$\sigma_r(\Delta,c_0)=\{\lambda:|\lambda-1|<1\}=III_1\sigma(\Delta,c_0)\cup III_2\sigma(\Delta,c_0).$$

Therefore, by Lemma 3.1, we have $III_1\sigma(\Delta,c_0)=\{1\}$. □

Theorem 3.3. *The following results hold*

(a) $\sigma_{ap}(\Delta, c_0) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\} \setminus \{1\}$.

(b) $\sigma_{\delta}(\Delta, c_0) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\}$.

(c) $\sigma_{co}(\Delta, c_0) = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\}$.

Proof. From Theorems 2.1, 2.2, 2.5 and 2.6 of Altay and Başar (2004), we have

$$I_3\sigma(\Delta, c_0) = II_3\sigma(\Delta, c_0) = III_3\sigma(\Delta, c_0) = \emptyset,$$
$$II_2\sigma(\Delta, c_0) = \{\lambda \in \mathbb{C} : |\lambda - 1| = 1\},$$
$$III_1\sigma(\Delta, c_0) \cup III_2\sigma(\Delta, c_0) = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\}.$$

Furthermore, $III_1\sigma(\Delta, c_0) = \{1\}$ by Corollary 3.2. Thus, the proof is obtained from the Table 2. □

The next corollary is an easy consequence of Proposition 2.1.

Corollary 3.4. *The following results hold:*

(a) $\sigma_{ap}(\Delta^*, \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\}$.

(b) $\sigma_{\delta}(\Delta^*, \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\} \setminus \{1\}$.

(c) [Theorem 2.3] $\sigma_p(\Delta^*, \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\}$.

3.2. Subdivision of the spectrum of Δ on c.

We deal with the subdivisions of the spectrum of the difference operator Δ over the sequence space c. Since the fine spectrum of the operator Δ on the space c can be derived by analogy to that of the space c_0, we omit the detail and give it without proof.

Lemma 3.5. $III_2\sigma(\Delta, c) = \{\{\lambda : |\lambda - 1| < 1\} \cup \{0\}\} / \{1\}$.

Corollary 3.6. $III_1\sigma(\Delta, c) = \{1\}$.

Theorem 3.7. *The following results hold*

(a) $\sigma_{ap}(\Delta, c) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\} \setminus \{1\}$.
(b) $\sigma_{\delta}(\Delta, c) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\}$.
(c) $\sigma_{co}(\Delta, c) = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\}$.

Proof. From Theorems 2.7, 2.8, 2.10 and 2.11 of Altay and Başar (2004), $I_3\sigma(\Delta, c) = II_3\sigma(\Delta, c) = III_3\sigma(\Delta, c) = \emptyset$, $II_2\sigma(\Delta, c) = II_2\sigma(\Delta, c) = \{\lambda \in \mathbb{C} : |\lambda - 1| = 1\}$ and $III_1\sigma(\Delta, c) \cup III_2\sigma(\Delta, c) = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\} \cup \{0\}$.

Moreover, $III_1\sigma(\Delta, c) = \{1\}$ by Corollary 3.6. Therefore, Table 2 leads us to the desired result. □

As a consequence of Proposition 2.1 we have

Corollary 3.8. *The following results hold*

(a) $\sigma_{ap}(\Delta^*, \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\}$.
(b) $\sigma_{\delta}(\Delta^*, \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\} \setminus \{1\}$.
(c) [Theorem 2.9] $\sigma_p(\Delta^*, \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\} \cup \{0\}$.

3.3. Subdivision of the spectrum of Δ on $\ell_p, (0 < p < \infty)$..

We give the subdivisions of the spectrum of the spectrum of the difference operator Δ over the sequence space ℓ_p, where $0 < p < \infty$.

Lemma 3.9. $1 \in III_1\sigma(\Delta, \ell_p), (0 < p < 1)$.

Proof. To verify the fact that

$$\Delta - I = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has bounded inverse, it is enough to show that $\Delta - I$ is bounded. Indeed, one can easily see for all $x=(x_n)\in \ell_1$ that

$$\|(\Delta - I)x\| = \sum_{n=1}^{\infty}|x_n| = \|x\|_1$$

which means that $\Delta - I$ is bounded. This completes the proof. □

Lemma 3.10. $III_2\sigma(\Delta,\ell_p)=\{\lambda\in\mathbb{C}:|\lambda-1|\leq 1\}\setminus\{1\}$, where $0<p<1$.

Proof. By Theorem 2.9 of Altay and Başar (2007), $\Delta-\lambda I\in III_1\cup III_2$.

Let $y=(y_n)\in\ell_\infty$. We desire to find $x=(x_n)\in\ell_p$ such that $(\Delta^*-\lambda I)x=y$. The solution of the system $(\Delta^*-\lambda I)x=y$ of the linear equations in the matrix form for x in terms of y gives the inverse matrix $(\Delta^*-\lambda I)^{-1}$. The nth row turns out to be

$$\begin{cases} 0 & ,k<n, \\ \dfrac{(1-\lambda)^n}{(1-\lambda)^{k+1}} & ,k\geq n. \end{cases}$$

Thus, we observe that

$$\left\|(\Delta-\lambda I)^{-1}\right\| = \sup_{n\in\mathbb{N}_1}\sum_{k=n}^{\infty}\frac{|1-\lambda|^n}{|1-\lambda|^{k+1}} = \sum_{j=1}^{\infty}\frac{1^n}{|1-\lambda|^j},$$

Which is convergent if $|1-\lambda|>1$. That is, if $|1-\lambda|>1$, then $\left(\Delta^*-\lambda I\right)^{-1}$ is surjective. Hence $\Delta-\lambda I$ has an unbounded inverse by II.311 Theorem of Goldberg (Goldberg (1966)). Therefore $\lambda\notin III_1\sigma(\Delta,\ell_p)$ and $III_2\sigma(\Delta,\ell_p)=\sigma_r(\Delta,\ell_p)\setminus III_1\sigma(\Delta,\ell_p)$.

This completes the proof. □

Theorem 3.11. *The following results hold*

(a) $\sigma_{ap}(\Delta, \ell_p) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\} \setminus \{1\}$.
(b) $\sigma_{\delta}(\Delta, \ell_p) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\}$.
(c) $\sigma_{co}(\Delta, \ell_p) = \begin{cases} \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\} & , \quad p \geq 1, \\ \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\} & , \quad 0 < p < 1. \end{cases}$

Proof. Let $p \geq 1$. Then, we have from Theorem 2.3 (Akhmedov and Başar (2006)) that $I_3\sigma(\Delta, \ell_p) = II_3\sigma(\Delta, \ell_p) = III_3\sigma(\Delta, \ell_p) = \emptyset$. We have $III_1\sigma(\Delta, \ell_p) \cup III_2\sigma(\Delta, \ell_p) = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\}$ from Theorem 2.7 (Akhmedov and Başar (2006)) and $III_1\sigma(\Delta, \ell_p) = \{1\}$ from Theorem 2.8 (Akhmedov and Başar (2006)). Thus, the proof follows from Table 2 for the case $1 \leq p < \infty$.

If $0 < p < 1$, then $I_3\sigma(\Delta, \ell_p) = II_3\sigma(\Delta, \ell_p) = III_3\sigma(\Delta, \ell_p) = \emptyset$ by Theorem 2.6 (Altay and Başar (2007)). We have $III_1\sigma(\Delta, \ell_p) \cup III_2\sigma(\Delta, \ell_p) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\}$ from Theorem 2.9 (Altay and Başar (2007)) and $III_1\sigma(\Delta, \ell_p) = \{1\}$ from Lemmas 3.9 and 3.10. Thus, Table 2 gives the desired result for the case $0 < p < 1$. □

Proposition 2.1 leads us to the following corollary.

Corollary 3.12. *Let* $p^{-1} + q^{-1} = 1$ and $p \geq 1$. Then, we have

(a) $\sigma_{ap}(\Delta^*, \ell_q) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\}$.
(b) $\sigma_{\delta}(\Delta^*, \ell_q) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\} \setminus \{1\}$.
(c) [Theorem 2.3] $\sigma_p(\Delta^*, \ell_q) = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\}$.

Corollary 3.13. *The following results hold*

(a) $\sigma_{ap}(\Delta^*, \ell_\infty) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\}$.
(b) $\sigma_{\delta}(\Delta^*, \ell_\infty) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\} \setminus \{1\}$.
(c) [Theorem 2.7] $\sigma_p(\Delta^*, \ell_\infty) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\}$.

3.4. Subdivision of the spectrum of Δ on bv_p, $(1 \le p < \infty)$.

In the present subsection, we give the subdivisions of the spectrum of the difference operator Δ over the sequence space bv_p, where $1 \le p < \infty$.

Theorem 3.14. *The following results hold*

(d) $\sigma_{ap}(\Delta, bv_p) = \{\lambda \in \mathbb{C} : |\lambda - 1| \le 1\} \setminus \{1\}$.

(e) $\sigma_{\delta}(\Delta, bv_p) = \{\lambda \in \mathbb{C} : |\lambda - 1| \le 1\}$.

(f) $\sigma_{co}(\Delta, bv_p) = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\}$.

Proof. We obtain that

$$III_3\sigma(\Delta, c) = II_3\sigma(\Delta, c) = III_3\sigma(\Delta, c) = \theta,$$
$$II_2\sigma(\Delta, bv_p) = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\},$$
$$III_1\sigma(\Delta, bv_p) = \{1\},$$
$$III_2\sigma(\Delta, bv_p) = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\}$$

From the Theorem 3.3, 3.8, 3.9 and 3.10 of Akhmedov and Başar (2007), respectively. Now the desired result is immediately obtained from Table 2. □

As a consequence of Proposition 2.1, we also have

Corollary 3.15. *The following results hold*

(a) $\sigma_{ap}(\Delta^*, bv_p^*) = \{\lambda \in \mathbb{C} : |\lambda - 1| \le 1\}$.

(b) $\sigma_{\delta}(\Delta^*, bv_p^*) = \{\lambda \in \mathbb{C} : |\lambda - 1| \le 1 \setminus \{1\}\}$.

(c) [Theorem 3.4] $\sigma_p(\Delta^*, bv_p^*) = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\}$.

4. CONCLUSION

There is a wide literature related with the spectrum and fine spectrum of certain linear operators represented by particular limitation matrices over some sequence spaces. Although the fine spectrum with respect to the Goldberg's classification of the operator Δ defined by a difference matrix over the sequence spaces ℓ_p, bv_p, c_0, c were respectively studied by Akhmedov and Başar (2006, 2007) and Altar and Başar (2004), in the present paper, the concepts of the approximate point spectrum, defect spectrum and compression spectrum are introduced and given the subdivisions of the spectrum of the difference matrix Δ over the sequence spaces $c_0, c, \ell_p (0 < p < \infty)$ and bv_p, $(1 < p < \infty)$, as the new subdivisions of spectrum. This is a development of the spectrum of an infinite matrix over a sequence space in the usual sense. By following the same way, it is natural that one can derive some new results from the known results via Table 2, in the usual sense.

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